Contact Geometry

Exercise sheet 2

Exercise 1.
On $S^3 \subset \mathbb{R}^4$ consider the 1-form

$$\alpha = x_1\, dy_1 - y_1\, dx_1 + \sqrt{2}(x_2\, dy_2 - y_2\, dx_2). \tag{1}$$

(a) Show that α is a contact form.

(b) Compute the Reeb flow of α. How many closed orbits has it?

(c) Show that $(S^3, \ker \alpha)$ is contactomorphic to (S^3, ξ_{st}).
Hint: Use Gray stability.

(d) A strict contactomorphism preserves the Reeb flow.

(e) Deduce that (S^3, α) and (S^3, α_{st}) are not strictly contactomorphic, that a contactomorphism does in general not preserve the Reeb flow, and that Gray stability does not hold true for contact forms.

Exercise 2.
Two contact structures ξ_0 and ξ_1 on a manifold M are called **isotopic** if there exist a family of diffeomorphisms $\psi_t\colon M \to M$ such that $\psi_0 = \mathrm{Id}$ and ψ_1 is a contactomorphism from (M, ξ_0) to (M, ξ_1).

(a) Consider on $\mathbb{R}^{2n+1}$ the contact forms

$$\alpha_0 = dz + \sum_{j=1}^{n} x_j\, dy_j \quad \text{and} \quad \alpha_0 = 2dz + \sum_{j=1}^{n} x_j\, dy_j - y_j\, dx_j.$$

Show that the induced contact structures are isotopic.

(b) Can the isotopy be chosen such that ψ_1 is a strict contactomorphism?

Exercise 3.
Prove Theorem 2.4 from the lecture.
Hint: Use the stereographic projection.

Exercise 4.

(a) Describe a contact structure on $\mathbb{R}P^3$.
Hint: Write $\mathbb{R}P^3$ as the unit cotangent bundle of a surface and use the canonical contact structure.

(b) Let M and N be diffeomorphic manifolds. Show that their unit cotangent bundles (with their canonical contact structures) are contactomorphic.

Exercise 5.
Let p and q be coprime integers and consider S^3 as the unit sphere in $\mathbb{C}^2$. Consider the $\mathbb{Z}_p$ action on S^3 generated by the diffeomorphism

$$(z, w) \longmapsto (e^{2\pi i/p} z, e^{2\pi i q/p} w).$$

(a) Show that this group action is free and thus the quotient is a smooth 3-manifold. We call that manifold the **lens space** $L(p, q)$.

(b) Show that $L(p, q)$ admits a contact structure.

(c) Let (M, ξ) be a contact manifold and let M' be a cover of M. Then M' admits a contact structure.

Bonus exercise.
Prove Lemma 2.10 from the lecture.

This sheet will be discussed on Thursday 9.11. and should be solved by then.